ON THE BEHAVIOR OF INTEGRAL CURVES IN THE NEIGHBORHOOD OF A PERIODIC MOTION

(O POVEDENII INTEGRAL'NYKH KRIVYKH V OKRESTNOSTI PERIODICHESKOGO DVIZHENIIA)

PMM Vol.25, No.2, 1961, pp. 303-313

V. I. ZUBOV
(Leningrad)

(Received October 17, 1960)

This article contains a set of rules which make it possible to distinguish between the qualitative pictures of the behavior of integral curves in the neighborhood of a periodic solution in the plane.

1. Statement of the problem. Basic definitions. Let us consider the system

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y) \tag{1.1}$$

We shall assume that the system (1.1) has the following properties:

a) the functions $f_i(x, y)$ are given in some region G of the xy-plane, they are real, continuous and twice differentiable with respect to their arguments;

b) there exist two differentiable real functions

$$x = \varphi_1(t), \quad y = \varphi_2(t) \tag{1.2}$$

periodic in t of period 2π, which constitute a solution of (1.1). The graph of this solution lies entirely in G.

It is known [1] that the periodic solutions fall, in relation to the structure of their neighborhoods, into two classes: the isolated ones, and the non-isolated ones.

Definition 1.1. The periodic solution (1.2) of the system (1.1) is said to be isolated if there exists a small enough δ-neighborhood $S(M, \delta) \subset G$ of the set M, which does not contain a graph of any other periodic solution of (1.1).

The isolated periodic solutions of the system (1.1) are called limit cycles. One distinguishes between three types of limit cycles according to the behavior of integral curves in their neighborhoods.

We shall denote by $\rho((x, y), M)$ the distance of the point (x, y) from the set M.

Definition 1.2. A limit cycle is said to be:

1) stable if there exists a small enough neighborhood $S(M, \delta) \subset G$ such that all integral curves of the system (1.1) which begin in $S(M, \delta)$ come arbitrarily close to M as $t \to \infty$. In other words, if $(x_0, y_0) \in S\, M, \delta)$, then the quantity $\rho((x, y), M) \to 0$ as $t \to \infty$, where

$$x = x(t, x_0, y_0), \qquad y = y(t, x_0, y_0) \tag{1.3}$$

is a solution of the system (1.1) whose graph passes through the point $(x_0, y_0($ when $t = 0$;

2) unstable, if there exists a small enough neighborhood $S(M, \delta)$ such that when $(x_0, y_0) \in S(M, \delta)$, then $\rho((x, y), M) \to 0$ when $t \to -\infty$;

3) semistable, if there exists a small enough neighborhood $S(M, \delta)$ which is divided by M into two regions S_1 and S_2 such that $\rho((x,y),M) \to 0$ when $t \to \infty$, $(x_0, y_0) \in S_1$ and $\rho((x, y), M) \to 0$ when $t \to -\infty$, $(x_0, y_0) \in S_2$.

In connection with all possible pictures (configurations) which illustrate the behavior of the integral curves in the neighborhood of a periodic solution (1.2), there arises the question on the stability in the Liapunov sense [2] of this periodic solution or on its conditional stability in the Liapunov sense. The problem of the present article is the formulation of criteria which will permit one to distinguish between various types of the qualitative behavior of integral curves, and also to determine the stability or instability, in the Liapunov sense, of the periodic solution (1.2).

2. The fundamental form of the equations of motion. Let us draw through each point of the graph M of the periodic solution (1.2) a normal, and let us take such a small neighborhood $S(M, \delta) \subset G$ that the segment of the normals contained in this neighborhood which correspond to distinct points of M will not intersect. Furthermore, the segment of the normal passing through the point $(\phi_1(t), \phi_2(t))$ which is contained in $S(M, \delta)$ will be denoted by N_t. By a theorem on continuity with respect to initial conditions, one can find a point $(x_0, y_0) \in N_0$ such that the solution (1.3) of the system (1.1) will lie in $S(M, \delta)$ when $t \in [-T, T]$, where $T > 0$ is chosen arbitrarily. Let us keep fixed the chosen point (x_0, y_0). We construct the segment of the normal N_t which corresponds to a small enough value $t > 0$. We denote by

$$\tau = \tau(t) \tag{2.1}$$

the first instant of the intersection of the graph of the solution (1.3) corresponding to the fixed initial point x_0, y_0 with the normal N_1 when t moves in the positive direction $t \geqslant 0$. We introduce into consideration the functions

$$z_1(t) = x(\tau(t), x_0, y_0) - \varphi_1(t), \qquad z_2(t) = y(\tau(t), x_0, y_0) - \varphi_2(t) \tag{2.2}$$

Now we consider the function

$$H.(z_1, z_2, t) = z_1 f_1(t) + z_2 f_2(t), \qquad (f_i(t) = f_i(\varphi_1(t), \varphi_2(t))) \tag{2.3}$$

Since $z_1(t)$, $z_2(t)$ is a vector colinear with $\mathbf{N}_t$, we have

$$H(z_1(t),\ z_2(t)) \equiv 0$$

Let us construct the differential equations whose solutions are the functions (2.1) and (2.2):

$$\frac{dz_1}{dt} = \frac{d\tau}{dt}\,\frac{dx(\tau(t), x_0, y_0)}{d\tau} = \frac{d\varphi_1}{dt}, \quad \frac{dz_2}{dt} = \frac{d\tau}{dt}\,\frac{dy(\tau(t), x_0, y_0)}{d\tau} - \frac{d\varphi_2}{dt} \tag{2.4}$$

$$\frac{d}{dt} H(z_1(t), z_2(t)) = \frac{dz_1(t)}{dt} f_1(t) + \frac{dz_2(t)}{dt} f_2(t) + z_1 \frac{df_1(t)}{dt} + z_2 \frac{df_2(t)}{dt} \equiv 0 \tag{2.5}$$

Making use of (1.1), (2.2), (2.4) and (2.5), we find

$$\frac{d\tau}{dt} = \frac{f_1^2(t) + f_2^2(t) - z_1 df_1(t)/dt - z_2 df_2(t)/dt}{f_1(t) f_1(z_1 + \varphi_1, z_2 + \varphi_2) + f_2(t) f_2(z_1 + \varphi_1, z_2 + \varphi_2)} \tag{2.6}$$

$$\frac{dz_1}{dt} = \frac{d\tau}{dt} f_1(z_1 + \varphi_1, z_2 + \varphi_2) - f_1(t),$$

$$\frac{dz_2}{dt} = \frac{d\tau}{dt} f_2(z_1 + \varphi_1, z_2 + \varphi_2) - f_2(t) \tag{2.7}$$

The relations (2.6) and (2.7) which we have found for the specific functions (2.1) and (2.2) can be treated as a system of differential equations for the determination of the functions τ, z_1 and z_2. This system has a number of properties which are useful for solving the stated problem.

Lemma 2.1. The function $H(z_1, z_2, t)$ defined by the relation (2.3) is an integral of the system (2.7).

Corollary. Let us consider the solution of the system (2.7)

$$z_i = z_i(t, z_1^\circ, z_2^\circ, t_0) \qquad (i = 1,2) \tag{2.8}$$

and the solution of Equation (2.6)

$$\tau = \tau(t, z_1^\circ, z_2^\circ, t_0) \tag{2.9}$$

determined by the initial conditions

$$z_i = z_i^\circ \text{ when } t = t_0, \qquad \tau = t_0 \text{ when } t = t_0.$$

By Lemma 2.1. the function H, evaluated on the solution (2.8) of the system (2.7), remains constant. Let us give a geometric interpretation of this situation.

Through the points of the graph M let us draw directed lines, so that the line, with direction ν_t, passing through the point $\phi_1(t)$, $\phi_2(t)$ makes an angle with the tangent whose cosine is given by the formula

$$\cos\psi_t = \frac{H(z_1, z_2, t)}{\sqrt{f_1^2(t) + f_2^2(t)}\sqrt{z_1^2 + z_2^2}}$$

where z_1 and z_2 are determined by (2.8).

One can assert that the solution (2.8) of the system (2.7) determines the solution

$$\begin{aligned} x(\tau - t_0, x_0, y_0) &= z_1(t, z_1^\circ, z_2^\circ, t_0) + \varphi_1(t) \\ y(\tau - t_0, x_0, y_0) &= z_2(t, z_1^\circ, z_2^\circ, t_0) + \varphi_2(t) \end{aligned} \tag{2.10}$$

of the system (1.1) with the initial conditions $x_0 = z_1^\circ + \phi_1(t_0)$, $y_0 = z_2^\circ + \phi_2(t_0)$ when $t = t_0$.

Indeed, differentiating both sides of Equations (2.10) with respect to t, and making use of Equations (2.6) and (2.7), we find that the functions x and y, as functions of the argument $\tau - t_0$, satisfy the system (1.1). Hereby, the solution (2.10) with $t = t_0$ passes through the point (x_0, y_0) which lies on the direction ν_{t_0}, and at the instant t passes through a point lying on the direction ν_1.

The above-given lemma makes it possible to lower the order of the system (2.6), (2.7) by utilizing its integral H.

Let us introduce new unknown functions by means of the formulas

$$\xi = z_1 f_2(t) - z_2 f_1(t), \qquad \eta = z_1 f_1(t) + z_2 f_2(t) \tag{2.11}$$

Inverting this transformation, we obtain

$$z_1 = \frac{\xi f_2(t) + \eta f_1(t)}{f_1^2(t) + f_2^2(t)}, \quad z_2 = \frac{-\xi f_1(t) + \eta f_2(t)}{f_1^2(t) + f_2^2(t)} \tag{2.12}$$

Differentiating (2.11) with the aid of (2.6) and (2.7), we find

$$\frac{d\xi}{dt} = \frac{d\tau}{dt}[f_1(z_1 + \varphi_1(t), z_2 + \varphi_2(t)) f_2(t) - f_2(z_1 + \varphi_1(t), z_2 + \varphi_2(t)) f_1(t)] + \frac{z_1 df_2(t)}{dt} - \frac{z_2 df_1(t)}{dt} \tag{2.13}$$

$$\frac{d\eta}{dt} = 0$$

Bearing in mind that the quantity η is constant, and taking into account the behavior of the solutions of the system (1.1) which start on N_0, one can set $\eta \equiv 0$, which does not restrict the generality of the problem stated in Section 1.

Keeping this in mind, we eliminate the functions z_1 and z_2, defined by Equations (2.12), from Equations (2.6) and (2.13).

Then we obtain

$$\frac{d\tau}{dt} = \frac{f_1^2(t) + f_2^2(t) - (a_1^\circ df_1(t)\, dt + a_2^\circ df_2(t) / dt)\ \xi}{f_1(t) f_1(a_1^\circ\xi + \varphi_1, a_2^\circ\xi + \varphi_2) + f_2(t) f_2(a_1^\circ\xi + \varphi_1, a_2^\circ\xi + \varphi_2)} \tag{2.14}$$

$$\frac{d\xi}{dt} = [f_1(a_1^\circ\xi + \varphi_1(t), a_2^\circ\xi + \varphi_2(t)) f_2(t) - f_2(a_1^\circ\xi + \varphi_1(t), a_2^\circ\xi + \varphi_2(t)) f_1(t)] \times$$
$$\times \frac{[f_1^2(t) + f_2^2(t) - (a_1^\circ df_1(t) / dt + a_2^\circ df_2(t) / dt)\ \xi]}{[f_1(t) f_1(a_1^\circ\xi + \varphi_1, a_2^\circ\xi + \varphi_2) + f_2(t) f_2(a_1^\circ\xi + \varphi_1, a_2^\circ\xi + \varphi_2)]} +$$
$$+ \left[a_1^\circ \frac{df_2(t)}{dt} - a_2^\circ \frac{df_1(t)}{dt}\right]\xi \tag{2.15}$$

where

$$a_1^\circ = \frac{f_2(t)}{f_1^2(t) + f_2^2(t)}, \qquad a_2^\circ = -\frac{f_1(t)}{f_1^2(t) + f_2^2(t)}$$

Let us denote the right-hand side of (2.14) by $F(\xi, t)$ and the right-hand side of (2.15) by $G(\xi, t)$. Setting $\tau = \theta + t$ in Equation (2.14), we obtain the equation

$$d\theta / dt = F(\xi, t) - 1 \tag{2.16}$$

for the determination of the function θ.

The last equation, together with the equation $d\xi / dt = F(\xi, t)$, we shall call the fundamental form of the equations of motion.

Theorem 2.1. In order that the periodic solution of the system (1.1) be stable in the sense of Liapunov, it is necessary and sufficient that the zero solution $\theta = 0$, $\xi = 0$ of the system (2.15), (2.16) be stable in the sense of Liapunov.

Proof. Necessity. Suppose that the periodic solution $a = \phi_1(t)$, $y = \phi_2(t)$ of the system (1.1) is stable in the Liapunov sense, i.e. for every fixed t_0 and any given $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that when $\sqrt{[(x_0 - \phi_1(t_0))^2 + (y_0 - \phi_2(t_0))^2]} < \delta$ then $\sqrt{[(x - \phi_1(t))^2 + (y - \phi_2(t))^2]} < \epsilon$ if $t \geqslant t_0$, where x, y is a solution of the system (1.1) with the initial conditions x_0, y_0 when $t = t_0$.

Let us consider the solution

$$\xi = \xi(t, \xi_0, t_0), \qquad \theta = \theta(t, \xi_0, \theta_0, t_0) \tag{2.17}$$

of the system (2.15), (2.16) with the initial conditions ξ_0, θ_0 when $t = t_0$.

The solution (2.17) can be expressed in terms of the solution of the system (1.1) in the following way.

Let $\tau(t, x_0, y_0, t_0)$ be a single-valued continuous function whose values give the instant of intersection of the solution $x(t - t_0, x_0, y_0)$, $y(t - t_0, x_0, y_0)$ of the system (1.1) with the N_t-direction, under the condition that the initial point (x_0, y_0) lies on the direction N_{t_0}.

Then the functions

$$\xi = f_2(t)\,[x(\tau - t_0, x_0, y_0) - \varphi_1(t)] - f_1(t)\,[y(\tau - t_0, x_0, y_0) - \varphi_2(t)] \tag{2.18}$$
$$\theta = \theta_0 + \tau - t$$

will yield the solution of the system (2.15), (2.16) with the initial conditions

$$\xi = \xi_0 = f_2(t_0)\,[x_0 - \varphi_1(t_0)] - f_1(t_0)\,[y_0 - \varphi_2(t_0)], \ \theta = \theta_0 \quad \text{when } t = t_0 \tag{2.19}$$

Our immediate aim is the estimation of $\tau - t$.

Suppose that L is the length of the closed curve M. Let us divide M by means of the points $A_0, \ldots, A_{n-1}$ into n equal arcs. On the arc $[A_j, A_{j+1}]$ we take a point B_j such that a circle with center at B_j will pass through the points A_j and A_{j+1} $(A_n = A_0)$. Let

$$v = \inf_{t \in [0,2\pi]} \sqrt{f_1^2(t) + f_2^2(t)} \qquad (t \in [0, 2\pi])$$

If we now denote by t_j the time length of the arc $[A_j, A_{j+1}]$, then we will have

$$t_j \leqslant L/nv$$

We shall assign different positions to the point A_0 on the curve M. Then on the basis of the principle of choice [3], there will exist a lower boundary $\rho_0 > 0$ for the radii of all possible circles. Inside each of these circles we construct a concentric circle of radius $a < \rho_0$. One can select $a_0 > 0$ so small that no two of the circles of radius a_0 with centers at B_j will intersect for any fixed position of A_0. Here, a_0, ρ_0 and t_j will approach zero if $n \to \infty$.

Suppose that for the sake of definiteness $\tau < t$ on some interval. Selecting A_0 so that $B_0 \in N_t$, and taking a_0 in the nature of ϵ, we consider the instant of time $t + t^\circ$, where t° is the time length of the arc $[B_0, B_1]$. At the instant $t + t^\circ$ we will have

$$\sqrt{[x(t+t^\circ-t_0, x_0, y_0) - \varphi_1(t+t^\circ)]^2 + [y(t+t^\circ-t_0, x_0, y_0) - \varphi_2(t+t^\circ)]^2} < \alpha$$

But then, during the time interval $[t, t + t^\circ]$, the solution of the system (1.1) will intersect the direction N_t. Hence, $t - \tau \leqslant t^\circ \leqslant 2L/nv$. From this we find that in general

$$|\tau - t| < 2L/nv \qquad (2.20)$$

Next, one can show that the solution of the system (1.1)

$$x(t - t_0, x_0, y_0), \qquad y(t - t_0, x_0, y_0)$$

is uniformly continuous when $t \geqslant t_0$.

Indeed, because of its stability this solution is contained for $t \geqslant t_0$ in $S(M, \epsilon)$. For a sufficiently small ϵ, dx/dt and dy/dt are bounded in $S(M, \epsilon)$, when $t \geqslant t_0$, by the same number, which shows the correctness of the above assertion.

We now give bounds for the functions ξ and θ:

$$|\xi| \leqslant |f_2(t)||x(t-t_0, x_0, y_0) - \varphi_1(t)| + |f_1(t)||y(t-t_0, x_0, y_0) - \varphi_2(t)| +$$
$$+ |f_2(t)||x(\tau - t_0, x_0, y_0) - x(t-t_0, x_0, y_0)| + |f_1(t)||y(\tau - t_0, x_0, y_0) -$$
$$- y(t-t_0, x_0, y_0)|, \quad |\theta| \leqslant |\theta_0| + |\tau - t|$$

From what has been established before, and from the last inequalities, it follows that for sufficiently small $|\xi_0|$ and $|\theta_0|$, the quantities

$|\xi|$ and $|\theta|$ will be arbitrarily small when $t \geqslant t_0$. This completes the proof of the necessity.

Sufficiency. Suppose the zero solution of the system (2.15), (2.16) is stable. We shall show that the periodic solution (1.2) of the system (1.1) is stable in the Liapunov sense.

Indeed, let us select the point $(x_0, y_0) \in S(M, \delta)$, where δ is a small enough positive number, and t_0 so that

$$\sqrt{[x_0 - \varphi_1(t_0)]^2 + [y_0 - \varphi_2(t_0)]^2} < \delta_1$$

Next we choose a $t_0°$ so that the directed segment $N_{t_0°}$ contains the point (x_0, y_0). We now construct the solution

$$\xi = \xi(t, \xi_0, t_0°), \qquad \theta = \tau(t, \xi_0, t_0°) - t$$

of the system (2.15), (2.16).

It is clear that if δ_1 is sufficiently small, then $|\xi_0|$, $|t_0 - t_0°|$ can be arbitrarily small, and hence the quantities

$$|x(\tau - t_0°, x_0, y_0) - \varphi_1(t)|, \qquad |y(\tau - t_0°, x_0, y_0) - \varphi_2(t)| \tag{2.21}$$

will also be small when $t \geqslant t_0°$.

From the uniform continuity of the functions $\phi_1(t)$ and $\phi_2(t)$ follows the smallness of the quantities

$$\varphi_i(\tau - t_0° + t_0) - \varphi_i(t) \quad \text{when } t \geqslant t_0°, \ i = 1,2 \tag{2.22}$$

In view of (2.21) and (2.22), we now find that the quantities

$$|x(t - t_0, x_0, y_0) - \varphi_1(t)|, \qquad |y(t - t_0, x_0, y_0) - \varphi_2(t)|$$

are arbitrarily small for $t \geqslant t_0$ if δ_1 is sufficiently small. This completes the proof of the sufficiency.

Note. The closed curve M divides its neighborhood $S(M, \delta)$ into two regions S_1 and S_2. One can ask the question of the stability, in the Liapunov sense, of the periodic solution (1.2) of the system (1.1) under the condition that the initial point (x_0, y_0) lies in one of the regions S_1 or S_2. The question on this type of conditional stability in the Liapunov sense can be reduced to the question on the stability (in the Liapunov sense) of the zero solution of the system (2.15), (2.16) under the condition that $\xi > 0$ or $\xi < 0$. The validity of this assertion follows from the proof of Theorem 2.1.

Theorem 2.2. To every periodic solution of the system (1.1) contained in a small enough neighborhood of M, there corresponds a 2π-periodic solution of Equation (2.15). This equation (2.15) has no other periodic solutions which lie in a correspondingly small enough neighborhood of the point $\xi = 0$.

Proof. Suppose that in the neighborhood $S(M, \delta)$ of the set M there is contained the graph of a periodic solution $\psi_1(t)$, $\psi_2(t)$ of the system (1.1) of period T.

Let us denote by $(\psi_1{}^0, \psi_2{}^0)$ the point of intersection of the graph of this solution with the direction N_0, and let us set

$$\xi_0 = f_2(0)[\psi_1{}^0 - \varphi_1(0)] - f_1(0)[\psi_2{}^0 - \varphi_2(0)]$$

We shall show that $\xi = \xi(t, \xi_0, 0)$ is a 2π-periodic solution of Equation (2.15). We may write the equation for the quantity τ which corresponds to the periodic solution $\psi_1(t)$, $\psi_2(t)$ in the form

$$d\tau/dt = f(\psi_1(\tau), \quad \psi_2(\tau), \ t) \tag{2.23}$$

Equation (2.23) is obtained from (2.14) if one replaces ξ in it by

$$\xi = f_2(t)[\psi_1(\tau) - \varphi_1(t)] - f_1(t)[\psi_2(\tau) - \varphi_2(t)] \tag{2.24}$$

The right-hand side of Equation (2.23) is a periodic function in t of period 2π and in τ of period T. We shall denote by $\tau(t)$ the solution of Equation (2.23) with the initial condition $\tau = 0$ when $t = 0$.

From geometric considerations it is clear that $\tau(2\pi) = T$. By means of a simple verification, we can convince ourselves that the function $\tau(t + 2\pi) - T$ is a solution of (2.23) with the same initial conditions as those of $\tau(t)$. Because of the uniqueness theorem we have $\tau(t + 2\pi) = \tau(t) + T$.

The last relation shows that the function ξ, defined by Formula (2.24), is a periodic function of period 2π.

If one now assumes that Equation (2.15) has a periodic solution, then one has to admit two possibilities: either two distinct values of ξ coincide on N_0, or this does not happen. In the first case, ξ is a periodic solution, of period 2π, corresponding to a solution of (1.1); in the second case, ξ will describe a spiral motion, and hence it cannot be a periodic solution of the system (2.15).

3. Analytic case. Let us assume that the right-hand sides in the system (1.1) are analytic in x and in y in a small enough region of M.

Then Equations (2.15), (2.16) can be expressed as

$$\frac{d\theta}{dt} = \sum_{k=1}^{\infty} a_k(t)\,\xi^k, \qquad \frac{d\xi}{dt} = \sum_{k=1}^{\infty} b_k(t)\,\xi^k \tag{3.1}$$

where the series in (3.1) converge when $|\xi| < r$, $r > 0$.

Direct computations show that

$$b_1(t) = \frac{\partial f_1(\varphi_1(t),\ \varphi_2(t))}{\partial x} + \frac{\partial f_2\ (\varphi_1(t),\ \varphi_2(t))}{\partial y} + \\ + \frac{1}{2}\frac{\partial}{\partial t}\ln[f_1^2(t) + f_2^2(t)] \tag{3.2}$$

Let us set

$$G_1 = \int_0^{2\pi} b_1(t)\,dt = \int_0^{2\pi}\left[\frac{\partial f_1(\varphi_1(t),\ \varphi_2(t))}{\partial x} + \frac{\partial f_2(\varphi_1(t),\ \varphi_2(t))}{\partial y}\right]dt \tag{3.3}$$

It is known [4] that when $G_1 < 0$ the periodic solution (1.2) of the system (1.1) is a stable limit cycle, and, as a matter of fact, it is stable in the sense of Liapunov; if $G_1 > 0$, the solution (1.2) is an unstable limit cycle which is stable in the Liapunov sense when $t \to -\infty$.

Let us see what happens when $G_1 = 0$. We make a change of variables in (3.1) by setting

$$\xi = \eta \exp\int_0^t b_1(t)\,dt \tag{3.4}$$

Then we obtain

$$\frac{d\theta}{dt} = \sum_{k=1}^{\infty} a_k{}^0(t)\,\eta^k, \qquad \frac{d\eta}{dt} = \sum_{k=2}^{\infty} b_k{}^0(t)\,\eta^k \tag{3.5}$$

Next we look for a solution of the second equation in (3.5) of the form

$$\eta = c + g_2(t)c^2 + \ldots + g_k(t)c^k + \ldots \tag{3.6}$$

where $g_k(t)$ ($k = 2, 3, \ldots$) are periodic functions of period 2π which are still to be determined, while c is an arbitrary constant.

Substituting (3.6) into (3.5) and equating coefficients of equal powers of c, we obtain

$$dg_k/dt = r_k \quad (k = 2,\ 3, \ldots) \tag{3.7}$$

If the functions $g_2, \ldots, g_{k-1}$ can be shown to be periodic functions, then the function g_k will also be periodic provided that

$$\int_0^{2\pi} r_k dt = G_k = 0$$

Let us assume first that $G_m \neq 0$ for some $m \geqslant 2$. Let this m be kept fixed. We next look for a solution of the first equation of (3.5) in the form

$$\theta = \sum_{k=1}^{\infty} h_k(t)\,\eta^k \tag{3.8}$$

Substituting (3.8) into (3.5) and equating equal powers of η, we find

$$dh_k/dt = P_k \qquad (k = 1, 2, \ldots) \tag{3.9}$$

After $h_1, \ldots, h_{k-1}$ have been found, and if they turn out to be periodic, then h_k will also be periodic, provided

$$\int_0^{2\pi} P_k dt = F_k = 0$$

Let us assume that there exists a number $l \geqslant 1$ such that $F_l \neq 0$. Let this l be fixed.

Theorem 3.2. 1) If m is odd and $G_m < 0$, then the periodic solution (1.2) of the system (1.1) is a stable limit cycle. If, furthermore, $l + 1 > m$, then this periodic solution will be stable in the Liapunov sense, but it will be unstable in the Liapunov sense if $l + 1 \leqslant m$.

2) If m is odd and $G_m > 0$, then the periodic solution (1.2) of the system (1.1) is an unstable limit cycle. If, furthermore, $l + 1 > m$, then this periodic solution is stable in the Liapunov sense when $t \to -\infty$, but it is unstable in this sense if $l + 1 \leqslant m$.

3) If m is even, then the periodic solution (1.2) of the system (1.1) is a semistable limit cycle. Furthermore, if $l + 1 > m$, then this periodic solution is conditionally stable in the Liapunov sense in that direction in which the integral curves of the system (1.1) approach arbitrarily close to M. If, however, $l + 1 \leqslant m$, then this type of conditional stability does not occur.

Proof. Let us make the change of variables

$$\eta = Z + \sum_{k=2}^{m-1} g_k Z^k + (g_m - G_m t)\,Z^m, \qquad \theta = \sigma + \sum_{k=1}^{l-1} h_k \eta^k + (h_l - F_l t)\,\eta^l \tag{3.10}$$

Then we obtain for the determination of the functions Z and σ the following equations:

$$\dot{Z} = G_m Z^m + \sum_{k=m+1}^{\infty} C_k(t) Z^k, \qquad \dot{\sigma} = F_l Z^l + \sum_{k=l+1}^{\infty} d_k(t) Z^k \tag{3.11}$$

Without loss of generality we may assume that the behavior of the solutions is being studied for $Z > 0$.

If we first consider the system of the first approximation

$$d\sigma/dt = F_l Z^l, \qquad dZ/dt = G_m Z^m \tag{3.12}$$

then its direct integration leads to the formulas

$$Z = Z_0 \{1 + [1 - m] G_m Z_0^{m-1} t\}^{\frac{1}{1-m}}$$

$$\underset{l \neq m-1}{\sigma = \sigma_0} + \frac{F_l}{(1-m) G_m Z_0^{m-1}} \frac{(m-1) Z_0^l}{(m-l-1)} \{[1 + (1-m) G_m Z_0^{m-1} t]^{\frac{m-l-1}{m-1}} - 1\}$$

When $l = m - 1$

$$\sigma = \sigma_0 + \ln [1 + (1-m) G_m Z_0^{m-1} t] \frac{F_l}{(1-m) G_m Z_0^{m-1}}$$

From these formulas follows the validity of the assertions made in Theorem 3.1, since the effect of the dropped terms is unessential.

Indeed, setting $Z = Z_0$ when $t = 0$, we obtain from (3.11)

$$\sigma = \sigma_0 + \int_{Z_0}^{Z} Z^{l-m} \frac{F_l + \dots}{G_m + \dots} dZ \tag{3.13}$$

For all $Z \leqslant r_0$, where r_0 is a sufficiently small positive number, we will have the inequality

$$a \leqslant \frac{F_l + \dots}{G_m + \dots} \leqslant b \tag{3.14}$$

where a and b are constants such that $ab > 0$.

From (3.13) and (3.14) we have, when $Z \leqslant r_0$

$$\sigma_0 + \frac{a}{l-m+1}[Z^{l-m+1} - Z_0^{l-m+1}] \leqslant \sigma \leqslant \sigma_0 + \\ + \frac{b}{l-m+1}[Z^{l-m+1} - Z_0^{l-m+1}] \quad (3.15)$$

if $l - m + 1 \neq 0$.

If $l - m + 1 = 0$, we have

$$\sigma_0 + a \ln \frac{Z}{Z_0} \leqslant \sigma \leqslant \sigma_0 + b \ln \frac{Z}{Z_0} \quad (3.16)$$

The number r_0 can be chosen so small that when $Z \leqslant r_0$ we have

$$CZ^m \leqslant G_m Z^m + \ldots \leqslant dZ^m, \qquad cd > 0 \quad (3.17)$$

Integrating the terms in the inequality

$$CZ^m \leqslant \frac{dZ}{dt} \leqslant dZ^m$$

we obtain

$$Z_0\{1 + (1-m)\, dZ_0^{m-1}t\}^{\frac{1}{1-m}} \leqslant Z \leqslant Z_0\{1 + (1-m)\, cZ_0^{m-1}\}^{\frac{1}{1-m}} \quad (3.18)$$

The inequality (3.18) will be valid whenever $Z \leqslant r_0$. The inequalities (3.15), (3.16) and (3.18) lead at once to the proof of the theorem in the general case.

Theorem 3.2. If for every m, $G_m = 0$, then there exists a neighborhood of the periodic solution (1.2) of the system (1.1) such that through every point of this neighborhood there passes a periodic solution of the system (1.1).

Hereby it is true that if $F_l = 0$ for every l, then the periodic solution (1.2) is stable in the Liapunov sense.

If, however, there exists an l such that $F_l \neq 0$, then the periodic solution (1.2) will not be stable in the Liapunov sense.

Proof. If $G_m = 0$ when $m \geqslant 2$, then all terms of the series (3.6) are periodic functions. If

$$g_k = \int_0^t r_k dt$$

then the series (3.6) is convergent [5] when $|c| \leqslant c_0$. Hence, the series determines a periodic solution for every c with modulus $|c| \leqslant C_0$. Besides that, when $F_l = 0$, $l \geqslant 1$, we have the equation (3.8) in which $h_k(t)$ will be a periodic function of period 2π, and the series on the right-hand side will converge when $h \leqslant h_0$. It follows from this that every periodic solution of the system (1.1) which lies in a small enough neighborhood of M has the period 2π. If, however, $F_l \neq 0$ for some l, then, by Theorem 2.1, one can conclude that the periodic solution (1.2) of the system (1.1) is unstable in the Liapunov sense.

Theorem 3.3. If the right-hand terms in the system (1.1) are analytic functions of x and y in some neighborhood of M, then the periodic solution (1.2) of the system (1.1) is either a limit cycle, or through every point of some small enough neighborhood $S(M, \delta)$ there passes a periodic solution of the system (1.1).

The proof is a direct consequence of Theorems 3.1 and 3.2.

4. Remarks of general nature. In this section we shall make some remarks of a general nature without giving any proofs of our assertions.

4.1. If the right-hand terms in the system (1.1) are n times differentiable in their arguments, then the right-hand terms of Equations (2.15) and (2.16) will have the same property.

If one applies Taylor's formula for the representation of the right-hand sides, and then drops the remainder terms, then one obtains a system of equations, for the determination of θ and ξ, with polynomial right-hand sides (on the right-hand side will occur polynomials with periodic coefficients). If in the application of Theorem 3.1 to these polynomials it should turn out that $m < n$, or $m > n$, then all the conclusions of Theorem 2.1 remain valid for the original system (1.1).

4.2. The application of Theorem 3.1, more precisely, of the ideas contained in its proof, makes it possible to obtain a deeper insight than that afforded by the usual results for systems containing a small parameter, even then when the solution (1.2) and its period depend on this parameter.

4.3. The general case can be covered by considering the roots of the equation

$$\psi(\xi_0) = 0$$

where

$$\psi = \xi(2\pi;\ \xi_0,\ 0) - \xi_0, \qquad |\xi_0| < \delta, \qquad \delta > 0 \text{ -- is sufficiently small}$$

BIBLIOGRAPHY

1. Nemytskii, V.V. and Stepanov, V.V., *Kachestvennaia teoriia differentsial'nykh uravnenii (Qualitative Theory of Differential Equations)*. GITTL, 1949.

2. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvizheniia (General Problem on the Stability of Motion)*. GITTL, 1950.

3. Fikhtengol'ts, G.M., *Kurs differentsial'nogo i integral'nogo ischisleniia (Course in Differential and Integral Calculus)*, Vol. 1. GITTL, 1947.

4. Andronov, A.A., Vitt, A.A. and Khaikin, S.E., *Teoriia kolebanii (Theory of Oscillations)*. Fizmatgiz, 1959.

5. Poincaré, H., *O krivykh, opredeliaemykh differentsial'nymi uravneniiami (On Curves Determined by Means of Differential Equations)*. (Russian translation from the French). Gostoptekhizdat, 1947.

Translated by H.P.T.